

Quantum integrability of sigma models on AII and CII symmetric spaces.

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October 2002

Abstract

Exact massive S-matrices for two dimensional sigma models on symmetric spaces $SU(2N)/Sp(N)$ and $Sp(2P)/Sp(P) \times Sp(P)$ are conjectured. They are checked by comparison of perturbative and non perturbative TBA calculations of free energy in a strong external field. We find the mass spectrum of the models and calculate their exact mass gap.

1 Introduction

Recently renewed interest to two dimensional sigma models with and without topological theta terms on different coset and, in particular, symmetric and supersymmetric spaces was stimulated by their relevance for Integer Quantum Hall effect [1] - [6], and two dimensional fermionic systems with different types of symmetry and quenched disorder (see, for example, [7] - [10], [2]). This is the reason why any new exact solution of such models, being important by itself in general, can also find a practical application.

Essential part of symmetric space sigma models are exactly solvable, since were proved (or believed) to be integrable. Classical integrability of sigma models on any symmetric spaces G/H is known for a long time [11], but quantum integrability for symmetric space with H non simple may be destroyed by possible anomalies [12]. But even for H simple, when quantum integrability is undoubted, exact S-matrices are known not for all symmetric spaces. (For Cartan classification of symmetric spaces see for example [17]).

Recently [13] the list of known exact S-matrices for integrable sigma models on symmetric spaces was extended by new massive and massless S-matrices for spaces AI ($SU(N)/SO(N)$) and BDI ($O(2P)/O(P) \times O(P)$) without and with theta term. The conjectured S-matrices were checked by comparison of $T = 0$ TBA calculations of free energy of the system in strong external field with perturbative calculations [13], and also by $T \rightarrow \infty$ TBA extraction of UV central charge for these two models [14]. It is remarkable that BDI sigma model turns out to be quantum integrable inspite of possibility of anomalies.

There are other close relatives of the models considered by Fendley: sigma model on AII ($SU(2N)/Sp(N)$) is close to AI , and sigma model on CII ($Sp(2P)/Sp(P) \times Sp(P)$) is analogous to BDI sigma model - in both cases orthogonal (sub)group is changed to symplectic. We are going to show that the analogy extends also to S-matrix conjecture, providing one more example (CII) of quantum integrability of non supersymmetric sigma model on a factor group manifold with non simple invariant subgroup.

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The plan of the discussion is the following. The first section describes standard perturbation theory technique of free energy calculation in a strong external field. In the second section we conjecture our fundamental S-matrices for the models using some symmetry arguments in support of it, and show how the mass spectrum of the model follows from bootstrap. After that we calculate the same free energy using $T \rightarrow 0$ TBA technique based on the conjectured exact S-matrix. We show correspondence between perturbative and non perturbative results which confirms correctness of the conjectured S-matrices. Moreover, two expressions for the same quantity - the free energy - fixes the mass gap for the models exactly. We conclude by brief discussion of results and give some technical details for perturbative calculations and the S-matrices, like explicit form of projectors, in the Appendix.

2 Perturbative analysis

We are considering the Lagrangian

$$\mathcal{L}_0 = \frac{1}{\lambda^2} Tr\{\partial_\mu \Phi \partial_\mu \Phi^\dagger\}$$

where we introduced a matrix representation Φ for coset G/H elements. Important property of this Lagrangian is its global G invariance. We work with Lie algebraic current representation of Lagrangian:

$$\mathcal{L}_0 = -\frac{1}{\lambda^2} Tr\{(g^{-1} \partial_\mu g)(g^{-1} \partial_\mu g)\} \quad (1)$$

and consider representation of a coset group element as an exponentialization of its Lie algebra: $g = \exp(i \sum_I n_I E^I + i \sum_i n_i H_i)$, where H_i are generators of Cartan subalgebra of a coset space, and E^I - other its generators. Explicit form of the basis E_I we use for calculations one can find in the Appendix.

In order to check our conjectures about an S-matrix of the model (see the next section), we do, for today, a standard procedure of putting the system into a strong external field [15]. Strong field (and hence high energy limit) gives an opportunity to believe to perturbation theory, because of asymptotical freedom of the models. So we replace derivatives in (1) by "covariant" ones:

$$D_\mu g = \partial_\mu g - h \delta_{\mu 0} (Qg + gQ)$$

where h is a strength of the external field, which is chosen in the direction \vec{q} in the Cartan subalgebra of G : $Q = \sum_{i=1}^r q_i H_i = (qH)$. The lagrangian density (1) in the presence of the source becomes

$$\mathcal{L} = \mathcal{L}_0 + \frac{2h}{\lambda^2} Tr\{(g^{-1}Q + Qg^{-1}) \partial_0 g\} - \frac{2h^2}{\lambda^2} Tr\{Q^2 + g^{-1}QgQ\} \quad (2)$$

We are going to calculate the dependence of the free energy on the external field h : $\delta f(h) = f(h) - f(0)$ using perturbative calculation in the running coupling constant $\lambda(h)$. We will restrict ourselves by quadratic part of the euclidean lagrangian in the fields n_I , which turns out to be enough for our purposes. Some details of calculations one can find in Appendix and the result is

$$\mathcal{L} \simeq -\frac{4h^2}{\lambda^2} q^2 + \frac{1}{\lambda^2} \mathcal{L}'_0$$

where for both *AI*I and *CII* cases

$$\mathcal{L}'_0 = \frac{1}{2} \sum_{i \geq j=1}^N \{n_{ij}((\partial_\mu)^2 + h^2 M_{ij}^I(q))n_{ij}^* + m_{ij}((\partial_\mu)^2 + h^2 M_{ij}^{II}(q))m_{ij}^*\} \quad (3)$$

with mass matrices

$$\begin{aligned} M_{ij}^I(q) &= (q_i - q_j)^2 + (q_{i+N} - q_{j+N})^2 \\ M_{ij}^{II}(q) &= (q_i - q_{j+N})^2 + (q_{i+N} - q_j)^2 \end{aligned}$$

for *AI*I case and

$$M_{ij}^I(q) = (q_i - q_{j+P})^2, \quad M_{ij}^{II}(q) = (q_i + q_{j+P})^2$$

for *CII* case. (Here in (3) there is no need in complex conjugation * in *CII* case.)

Lets point out that at this level some of the fields n_I decoupled, and we wrote only those which enter in an h dependent maner. At the tree level one has $\delta f(h)_0 = -\frac{4h^2}{\lambda^2}q^2$, $q^2 = \sum_{i=1}^{2N(2P)} q_i^2$. Free energy at the one loop level is just properly regularized $\frac{1}{2} \sum_{i,j} \ln \det((\partial_\mu)^2 + h^2 M_{ij}(q))$. Using standard dimensional regularization $\varepsilon = d - 2$ one finds

$$\delta f(h)_1 = -\frac{h^2 \beta_1}{2\pi\varepsilon} q^2 + \frac{h^2}{4\pi} \sum_I \sum_{i \geq j=1} M_{ij}^I(q) \left[1 - \gamma_E + \ln 4\pi - \ln \left(\frac{h^2}{\mu^2} M_{ij}^I(q) \right) \right]$$

where μ is a mass parameter of dimensional regularization, and $\beta_1 q^2 = \sum_I \sum_{i \geq j=1} M_{ij}^I(q)$. After some algebra one can find that $\beta_1 = 2N$ for *AI*I case, and $\beta_1 = 2P + 1$ for *CII* case. Here we use the condition $\sum_1^{2N(2P)} q_i = 0$, which is necessary in *AI*I case, and just will correspond to our concrete choice of the external field in *CII* case (see below).

The point is that the quantity $\delta f(h)$ is renormalization group invariant when λ runs with μ , so we can set $\mu = h$ and use the results of β -function calculations (without external field), done for almost all symmetric spaces [21] up to three loops. We need the result up to two loops:

$$h \frac{\partial}{\partial h} (\lambda^2) = -\frac{1}{8\pi} (\beta_1 \lambda^4 + \beta_2 \lambda^6) - O(\lambda^8) \quad (4)$$

where $\beta_2 = 2N(N - 1)$ for *AI*I and $\beta_2 = P(2P + 1)$ for *CII*, and β_1 is the same as above, since our calculation reproduced the correct form of one loop beta function. So adding necessary counterterm to lagrangian we get the following expression for the free energy

$$\delta f(h) = -\frac{4h^2}{\lambda(h)^2} q^2 - \frac{h^2}{4\pi} \sum_I \sum_{i \geq j=1} M_{ij}^I(q) [\ln M_{ij}^I(q) - 1] + O(\lambda^2) \quad (5)$$

One can solve equation (4)

$$\frac{1}{\lambda^2(h)} = \beta_1 \ln \frac{h}{\Lambda_{MS}} + \frac{\beta_2}{\beta_1} \ln \ln \frac{h}{\Lambda_{MS}} + O\left(\frac{1}{\ln \frac{h}{\Lambda_{MS}}}\right)$$

where Λ_{MS} is the cutoff parameter of minimal subtraction scheme, and substitute it into (5):

$$\delta f(h) = -4h^2 q^2 \beta_1 \left(\ln \frac{h}{\Lambda_{MS}} + \frac{\beta_2}{\beta_1^2} \ln \ln \frac{h}{\Lambda_{MS}} + c \right) \quad (6)$$

$$c = \sum_J \sum_{i \geq j=1} M_{ij}^J(q) [\ln M_{ij}^J(q) - 1] \quad (7)$$

This expression will be used for comparison with result of free energy calculation by TBA based on exact S-matrix, which we are going to present now.

3 Exact S-matrices

As in many other examples of quantum integrable models with higher rank Lie algebraic (actually Yangian) symmetries, one can expect that particles group into multiplets corresponding to irreducible representations of the symmetry. As we mentioned, there is a global G symmetry acting on the coset space G/H , so we assume the S-matrix is related to branching rules of decomposition of highest weight reps of G into irreducible reps of H . The fundamental S-matrix usually is related to the shortest highest weight reps of G . The shortest highest weight reps of both $SU(2N)$ and $Sp(2P)$, which appear in decomposition of their adjoint representation into irreps of $Sp(N)$ and $Sp(P) \times Sp(P)$ correspondingly, is representation μ_2 -second fundamental weight (antisymmetric) representation of $SU(2N)$ and $Sp(2P)$ (see, e.g. [22]). This gives rise to our conjecture: the fundamental S-matrix of AII and CII symmetric space sigma models are described by rational $\mu_2 \times \mu_2$ S-matrices of $SU(2N)$ and $Sp(2P)$ symmetry correspondingly. As it is well known, Lie algebraic symmetry with crossing and unitarity does not fix S-matrix completely. The remaining so called CDD ambiguity is very important, since it in particular may change the pole structure of the S-matrix, i.e. defines bound states and spectrum of the model. This CDD ambiguity should be resolved using any kind of arguments, e.g. physically required coincidence of the S-matrix to a some known one, at a specific value of one of its parameters. There are two types of rational S-matrices of general series of Lie algebraic symmetries. Gross-Neveu like S-matrices have additional CDD factors with poles which, through the bootstrap, lead to a set of massive multiplets corresponding to all highest fundamental weights reps, while sigma model like S-matrices usually are "minimal" (have no poles in the physical strip of rapidity) and have no these CDD factors. As it was conjectured and confirmed by different checks [13][14], sigma models on the symmetric spaces AI and BDI , are similar rather to Gross Neveu models, since they have bound state coming from CDD poles. The same happens in our case.

The fundamental S-matrix of AII sigma model has the form

$$S_{\mu_2, \mu_2}(\theta) = X(x) S_{\min}(x) \left(\mathcal{P}_{2\mu_2} + \frac{\Delta + x}{\Delta - x} \mathcal{P}_{\mu_3 + \mu_1} + \frac{\Delta + x}{\Delta - x} \frac{2\Delta + x}{2\Delta - x} \mathcal{P}_{\mu_4} \right) \quad (8)$$

with

$$\begin{aligned} X(x) &= -\frac{\sinh \frac{1}{2}(\theta + 4\pi i/2N)}{\sinh \frac{1}{2}(\theta - 4\pi i/2N)} = \frac{\sin \pi(2\Delta + x)}{\sin \pi(2\Delta - x)}, \\ S_{\min}(x) &= \frac{\Gamma(1-x) \Gamma(\Delta+x) \Gamma(1-\Delta-x) \Gamma(2\Delta+x)}{\Gamma(1+x) \Gamma(\Delta-x) \Gamma(1-\Delta+x) \Gamma(2\Delta-x)} \end{aligned} \quad (9)$$

where $x = \frac{\theta}{2\pi i}$, $\Delta = \frac{1}{2N}$, and \mathcal{P}_ω -projector on a rep with highest weight ω . Explicit form of the projectors one can find in Appendix.

For the CII sigma model we conjecture the following form of the fundamental S-matrix

$$\begin{aligned} S_{\mu_2, \mu_2}(x) &= X(x) S_{\min}(x) \left(\mathcal{P}_{2\mu_2} + \frac{\frac{1}{2} - \Delta + x}{\frac{1}{2} - \Delta - x} \mathcal{P}_{2\mu_1} + \frac{\frac{1}{2} - \Delta + x}{\frac{1}{2} - \Delta - x} \frac{\frac{1}{2} + x}{\frac{1}{2} - x} \mathcal{P}_0 \right. \\ &\quad \left. + \frac{\Delta + x}{\Delta - x} \mathcal{P}_{\mu_1 + \mu_3} + \frac{\Delta + x}{\Delta - x} \frac{\frac{1}{2} - \Delta + x}{\frac{1}{2} - \Delta - x} \mathcal{P}_{\mu_2} + \frac{\Delta + x}{\Delta - x} \frac{2\Delta + x}{2\Delta - x} \mathcal{P}_{\mu_4} \right) \\ S_{\min}(x) &= \frac{\Delta + x}{\Delta - x} \frac{\Gamma(-\Delta - x) \Gamma(2\Delta + x) \Gamma(1-x) \Gamma(\Delta + x)}{\Gamma(-\Delta + x) \Gamma(2\Delta - x) \Gamma(1+x) \Gamma(\Delta - x)} \times \end{aligned} \quad (10)$$

$$X(x) = \frac{\Gamma(1+x)\Gamma(\frac{1}{2}+\Delta-x)\Gamma(\frac{1}{2}-\Delta+x)\Gamma(\frac{1}{2}+2\Delta-x)}{\Gamma(1-x)\Gamma(\frac{1}{2}+\Delta+x)\Gamma(\frac{1}{2}-\Delta-x)\Gamma(\frac{1}{2}+2\Delta+x)} \times \frac{\sin \pi(x+2\Delta)\sin \pi(x+\frac{1}{2}-2\Delta)}{\sin \pi(x-2\Delta)\sin \pi(x-\frac{1}{2}+2\Delta)}$$

where $\Delta = \frac{1}{2(2P+1)}$.

In both cases the product S_{\min} and parenthesis is the minimal unitary and crossing symmetric S-matrices of $SU(2N)$ and $Sp(2P)$ symmetry, fused from corresponding elementary vector representation S-matrices [18][19][20]. They have no poles on the physical strip of rapidities θ . Additional CDD factors in both cases provide the only pole (and hence a bound state particle) in the \mathcal{P}_{μ_4} channels (in Sp case there is also cross channel pole). It is clear that S-matrix describing the scattering of particle from fundamental μ_2 multiplet on the particle from μ_4 multiplet may be obtained by fusion and will give a pole in μ_6 projector, and so on. In this way we get a spectrum for both models described by μ_{2k} multiplets, ($k = 1, \dots, N-1$ for AII case and $k = 1, \dots, P$ for CII case). Mass spectrum can be written as $m_k = M \sin\left(\pi \frac{k}{N}\right)$ for AII , and $m_k = M \sin\left(\pi \frac{2k}{2P+1}\right)$ for CII , where M is a mass scale.

As it sometimes happens in series of higher rank symmetric integrable models, in their lowest rank cases they often coincide with some other series (for example, lowest rank thermally perturbed WD_n CFT $n = 2$ are just parafermions with a proper perturbation). A remarkable hint for integrability of the models we are considering here, we get from the fact that $SU(4)/Sp(2)$ is isomorphic to $SO(6)/SO(5)$. It means that at $N = 2$ our AII S-matrix should have the well known form of $O(6)$ sigma model. In the same way for CII case $Sp(2)/Sp(1) \times Sp(1) \sim SO(5)/SU(2) \times SU(2) \sim SO(5)/SO(4)$ - it is the $O(5)$ sigma model. Let's recall that $O(K)$ sigma model has only one (vector) multiplet of particles in the spectrum (no bound states) with the following S-matrix

$$S^{O(K)}(\theta) = \frac{\Gamma(1-x)\Gamma(\frac{1}{2}+x)}{\Gamma(1+x)\Gamma(\frac{1}{2}-x)} \frac{\Gamma(x+\frac{1}{K-2})\Gamma(\frac{1}{2}+\frac{1}{K-2}-x)}{\Gamma(-x+\frac{1}{K-2})\Gamma(\frac{1}{2}+\frac{1}{K-2}+x)} \left(\mathcal{P}_S + \frac{x+\frac{1}{K-2}}{x-\frac{1}{K-2}} \mathcal{P}_A + \frac{x+\frac{1}{K-2}}{x-\frac{1}{K-2}} \frac{x+\frac{1}{2}}{x-\frac{1}{2}} \mathcal{P}_0 \right)$$

After some Γ function algebra one can see that $K = 6$ case really coincides with (8) at $N = 2$, with corresponding representations mapping $\mathcal{P}_S \rightarrow \mathcal{P}_{2\mu_2}, \mathcal{P}_A \rightarrow \mathcal{P}_{\mu_1+\mu_3}, \mathcal{P}_0 \rightarrow \mathcal{P}_{\mu_4}$. In the same way one can check that $K = 5$ S-matrix is the same as (10) at $P = 1$. In this case representation correspondence has the form $\mathcal{P}_S \rightarrow \mathcal{P}_{2\mu_2}, \mathcal{P}_A \rightarrow \mathcal{P}_{2\mu_1}, \mathcal{P}_0 \rightarrow \mathcal{P}_0$. (Projectors $\mathcal{P}_{\mu_1+\mu_3}, \mathcal{P}_{\mu_2}, \mathcal{P}_{\mu_4}$ are absent in this case.).

4 TBA calculation of the free energy

The main point in $T = 0$ TBA analysis of our models in external field is based on a skill to choose external field in such a way, that TBA system will be the simplest, i.e. the ground state will contain the minimal number of particles generated from vacuum by external field. The fact that the field is strong gives a basis for the assumption that only particles with maximal charge will

be generated by external field. As we said the fundamental S-matrices for both *AII* and *CII* models are rank two antisymmetric tensors a_{ij} and an external field from the Cartan subalgebra of G , $A = \text{diag}_{2N,2N}\{A_1, \dots, A_{2N}\}$, acts on them as $Aa_{ij} = (A_i + A_j)a_{ij}$.

For the *AII* case we chose the field in the form

$$A = \frac{1}{\sqrt{8}} \text{diag}_{2N,2N}\{1, 1, -\frac{1}{N-1}, \dots, -\frac{1}{N-1}\} \quad (11)$$

(We work with the same normalization for fields and charges, and the meaning of our normalization choice $\frac{1}{\sqrt{8}}$ will be clear below). Then the ground state will contain only particles of the type a_{12} , since they have the maximal charge 2. One can see using the explicit form of projectors (see Appendix) that the scattering process $a_{12} + a_{12} \rightarrow a_{12} + a_{12}$ takes place only in the $P_{2\mu_2}$ channel and the S-matrix for it is a prefactor before the parenthesis in (8).

The situation is more complicated in *CII* case. We chose

$$A = \frac{1}{\sqrt{8}} \text{diag}_{4P,4P}\{1, -1, 0, \dots, 0, -1, 1, 0, \dots, 0\} \quad (12)$$

with non zero elements on the places $1, 2, 2P+1, 2P+2$. In principle any combination of particles which is $O(2)$ invariant and has a maximal charge in the field A , can serve as a representative of the ground state. One can see that the combination $d = a_{1,P+2} - a_{2,P+1} + a_{1,P+1} - a_{2,P+2}$ has the maximal charge 2. In addition this particle has two important properties, which one can find analysing the projectors (see Appendix): firstly, $(dd)_{ij} = 0$, and hence $P_{\mu_2}, P_{2\mu_1}, P_0$ are zero for the scattering of d on itself, and, secondly, this particle scattering on itself does not produce other particles and amplitude of the scattering is the coefficient before the projector $P_{2\mu_2}$ - the prefactor before the parenthesis in (10).

So in both *AII* and *CII* cases, with the choice of external field we described above, we have one particle in the ground state. Following standard technology of thermodynamic Bethe ansatz (TBA), one can get the TBA equation for the so called dressed energies of the particles

$$\varepsilon(\theta) = h - m \cosh \theta + \int_{-B}^B d\theta' \phi(\theta - \theta') \varepsilon(\theta') \quad (13)$$

in terms of which free energy as a function of external field is

$$F(h) - F(0) = -\frac{m}{2\pi} \int_{-B}^B d\theta \cosh(\theta) \varepsilon(\theta) \quad (14)$$

Here B is a function of h/m determined by the boundary condition $\varepsilon(\pm B) = 0$, and ϕ is a kernel defined by the S matrix $S(\theta) = X(\theta)S_{\min}(\theta)$ for the scattering of the particles:

$$\phi(\theta) = \frac{1}{2\pi i} \frac{d}{d\theta} \ln S(\theta)$$

The Wiener-Hopf method of solution of the integral equation of the form (13) in the limit $h/m \rightarrow \infty$, aimed to extraction of the free energy (14), gives an answer for it

$$F(h) - F(0) = -\frac{\kappa^2}{4} h^2 \left[\ln \frac{h}{m} + \left(s + \frac{1}{2}\right) \ln \ln \frac{h}{m} + c' + \dots \right] \quad (15)$$

if Fourier transform of the kernel $\hat{\phi}(\omega) = 1 - K(\omega)$ factorizes into $K(\omega) = \frac{1}{K_+(\omega)K_-(\omega)}$, where $K_{\pm}(\omega)$ are bounded and have no poles or zeros in the upper(lower) half plane and have an asymptotic for small ξ

$$K_+(i\xi) = \frac{\kappa}{\sqrt{\xi}} e^{-s\xi \ln \xi} (1 - b\xi + O(\xi^2)) \quad (16)$$

with some constants s , κ and b , and

$$c' = \ln \frac{\sqrt{2\pi\kappa}e^{-b}}{K_+(i)} - 1 + s(\gamma_E - 1 + \ln 8) \quad (17)$$

The detailed proof of this statement one can find in [23]. Calculation of the Fourier transform of the kernels gives

$$K(\omega) = 2e^{\pi|\omega|\Delta} \frac{\sinh(\pi|\omega|\Delta) \sinh(\pi\omega(1-2\Delta))}{\sinh(\pi\omega)} \quad (18)$$

for *AII* case and

$$K(\omega) = 2e^{\pi|\omega|\Delta} \frac{\sinh(\pi|\omega|\Delta) \cosh(\pi\omega(\frac{1}{2}-2\Delta))}{\cosh(\pi\omega/2)} \quad (19)$$

for *CII* case. By $|\omega|$ we mean here the function which has the following analitical continuation to the whole complex plane: $|\omega| \rightarrow \omega * \text{sign}(Re(\omega))$. Factorization of (18) may be done as

$$K_+(\omega) = \sqrt{-i \frac{\Delta(1-2\Delta)\omega}{2\pi}} e^{-i\omega\Delta \ln(-i\omega) + i\mu\omega} \frac{\Gamma(-i\omega\Delta) \Gamma(-i\omega(1-2\Delta))}{\Gamma(-i\omega)}$$

where the boundness of K_+ requires $\mu = \Delta \ln \Delta + (1-2\Delta) \ln(1-2\Delta)$, which leads to the asymptotic behaviour of the type (16) with the constants

$$s = -\Delta, \quad \kappa = \frac{1}{\sqrt{2\pi\Delta(1-2\Delta)}}, \quad b = \mu - \Delta\gamma_E$$

Kernel (19) factorizes as

$$K_+(\omega) = \sqrt{-i \frac{\Delta\omega}{2\pi}} e^{-i\omega\Delta \ln(-i\omega) + i\mu\omega} \frac{\Gamma(-i\omega\Delta) \Gamma(\frac{1}{2} - i\omega(\frac{1}{2} - 2\Delta))}{\Gamma(\frac{1}{2} - \frac{i\omega}{2})}$$

with $\mu = \Delta \ln \Delta + (\frac{1}{2}-2\Delta) \ln(\frac{1}{2}-2\Delta) + \frac{1}{2} \ln 2$. It has asymptotic (16) with the following constants:

$$s = -\Delta, \quad \kappa = \frac{1}{\sqrt{2\pi\Delta}}, \quad b = \mu + \gamma_E\Delta$$

5 Comparison of results

We are going now to compare the results of perturbative (6) and non perturbative (15) TBA calculation of the free energy. First of all, using the form of external fields (and hence the charges) we chosed (11),(12), one can see that prefactors of parenthesis coincide for both *AII* and *CII* cases. Although the normalization of charges is ambiguous total factor, the fact that in this prefactor we get in both cases the same dependence on N and on P , is highly nontrivial check of not only our S-matrix conjecture, but also of the conjecture about the particle content of the ground state. Second, even more impressive, check is the comparison of the coefficients before the subleading term $\ln \ln h$ in both formulas. One of them is defined by beta function coefficients, another - by purely exact S-matrix dependent TBA analysis. Again, they coincide for both *AII* and *CII* cases. Moreover, after we saw the coincidence of the leading and subleading terms in the limits $h \gg \Lambda_{MS}$, $h \gg m$, one can use different expressions (6) and (15) for the same quantity in order to fix the relation between mass scale m and the renormalization scheme parameter Λ_{MS} .

This comparison involves the constant terms c, c' in both expressions. In the leading order of big h one just has what is called exact mass gap for the models

$$\ln \frac{m}{\Lambda_{MS}} = c - c' + O(1/\ln h)$$

Using the form of the external field (11),(12) and the expressions for c, c' (7),(17), one can calculate the gap explicitly for both sigma models.

6 Discussion

We proposed fundamental S-matrices for two dimensional sigma models on AII and CII symmetric spaces. We checked them by comparison of perturbative and TBA calculations of free energy in a strong external field of a specific form and found the desired correspondence. In this check not all the particles of the conjectured spectrum have participated - only subsector of fundamental S-matrix was used. In this sense $T > 0$ TBA check based on extraction of UV central charge seems to be a more complete, since it involves all the particles of the spectrum. We hope to report on this soon [25].

The quantum integrability of AII sigma model was expected since the factorization subgroup is simple in this case, but the integrability of CII sigma model is a "surprise" in this sense, because one might expect anomalies for non local current conservation [12]. It is clear that a deeper understanding of their integrability is desired from the point of view of conserved currents algebra and their symmetry. More exactly, the question remaining unclear is what kind of Yangian symmetry is responsible for integrability of sigma models on coset spaces. For today, the only known to us mathematically rigorous formulation of coset like symmetric Yangians are twisted Yangians, but they are known to be responsible for boundary integrability of sigma models [24], and hardly have something to do with quantum integrability without boundary.

7 Acknowledgments

The author is grateful to S.Elitzur for many helpful discussions and to P.Fendley for useful communications.

8 Appendix

8.1 Bilinear action

If we chose the symplectic form as $2N$ by $2N$ matrix of the following block form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, the basis (non orthonormal) for the generators of Lie algebra of AII symmetric space one can chose in the form of the following $2N$ by $2N$ matrices

$$\begin{aligned} E_{ij}^I &= (E_{ij} + E_{j+N,i+N}), \quad i \neq j \\ E_{ij}^{II} &= (E_{i,j+N} - E_{j,i+N}) \\ E_{ij}^{III} &= (E_{i+N,j} - E_{j+N,i}) \\ H_i &= (E_{ii} - E_{i+1,i+1} + E_{i+N,i+N} - E_{i+N+1,i+N+1}) \end{aligned}$$

where i, j in the first three lines are running from 1 to N , and in the last (Cartan) generators - i runs from 1 to $N - 1$. Here E_{ij} is $2N$ by $2N$ matrix with one non zero element equal to 1 and located at the position (i, j) . The motivation for this choice is clear: the first 3 types of generators with opposite choice of sign between two E belong to $Sp(N)$ since are of the form

$$\begin{pmatrix} a & b \\ c & -a^T \end{pmatrix}$$

required for Lie algebra of $Sp(N)$ with the symplectic form choice we made, where a, b, c are N by N matrices and b and c - symmetric. So, opposite choice of signs means that these generators are in orthogonal completion, i.e. in the coset algebra $su(2N)/sp(N)$. Condition of unitarity for general $g = \exp \left(i \sum_I n^I(x) E_I \right)$ now reads as $n_{ij}^{I*} = n_{ji}^I, n_{ij}^{II*} = n_{ji}^{III}, n_{ij}^{III*} = n_{ji}^{II}$. It is clear that n^{II}, n^{III} can be taken antisymmetric.

Generators of $sp(2P)/sp(P) \times sp(P)$ coset Lie algebra may be chosen in the form of $4P$ by $4P$ matrices, dividing them into 16 P by P block matrices. We define the action of the first invariant $sp(P)$ subalgebra in the four blocks $(1, 1), (1, 3), (3, 1), (3, 3)$ then the second $sp(P)$ acts in the blocks $(2, 2), (2, 4), (4, 2), (4, 4)$. We require the generators of the coset Lie algebra just to be zero in these eight blocks and get the following basis:

$$\begin{aligned} E_{ij}^I &= E_{i,j+P} - E_{j+3P,i+2P} \\ E_{ij}^{II} &= E_{i+P,j} - E_{j+2P,i+3P} \\ E_{ij}^{III} &= E_{i,j+3P} + E_{j+P,i+2P} \\ E_{ij}^{IV} &= E_{i+2P,j+P} + E_{j+3P,i} \end{aligned}$$

These are non zero in the remaining eight blocks, and are generators of $sp(2P)$. The reality condition leads to the requirement $n_I^T = n_{II}, n_{III} = n_{IV}$.

Calculation of bilinear terms may be done by expansion of exponent for coset group element up to the second order in fields and substitution of it into (2). For instance for \mathcal{L}_0 this leads to $\mathcal{L}_0 = -Tr(\lambda_a \lambda_b) \partial_\mu n_a \partial_\mu n_b$, where λ_a, n_a - a total set of Lie algebra generators and corresponding fields. Explicit calculation also shows that the term linear in \hbar gives total derivatives, and may be omitted. In the same way, terms containing fields for Cartan generators n_i drop out from the term proportional to \hbar^2 in (2). So fields n_i in both cases decouple and may be omitted. For the remaining fields we change notations and normalization: AII case $n_I \rightarrow n, n_{II} \rightarrow m$, CII case $n_I \rightarrow n, n_{III} \rightarrow m$, and get the actions (3).

8.2 Projectors

Here we present the projectors appearing in irreducible decomposition of tensor product of two antisymmetric representations. For projectors appearing in irrep decomposition of tensor product of two antisymmetric representations (highest weight μ_2) for $SU(2N)$ written in terms two antisymmetric tensors of rank 2 a_{ij} and b_{kl} , one can get using standard Yang tableau technique.

$$\begin{aligned} \mathcal{P}_{\mu_4} &= \frac{1}{6}(a_{ij}b_{kl} + a_{il}b_{jk} + a_{kl}b_{ij} + a_{jk}b_{il} - a_{ik}b_{jl} - a_{jl}b_{ik}) \\ \mathcal{P}_{2\mu_2} &= \frac{1}{6}(2a_{ij}b_{kl} + 2a_{kl}b_{ij} + a_{kj}b_{il} + a_{ik}b_{jl} + a_{jl}b_{ik} + a_{il}b_{kj}) \\ \mathcal{P}_{\mu_1+\mu_3} &= \frac{1}{2}(a_{ij}b_{kl} - a_{kl}b_{ij}) \end{aligned} \tag{20}$$

With more work one can get also projectors appearing in irrep decomposition of tensor product of two antisymmetric representations (highest weight μ_2) for $Sp(N)$ written in terms two antisymmetric rank 2 tensors a_{ij} and b_{kl} traceless in the sense that $\sum_{i,j} a_{ij} \sigma_{ij} = 0$, where we chose for symplectic form $\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ with four $N \times N$ block matrices.

$$\begin{aligned}
\mathcal{P}_{\mu_4} &= \frac{1}{6} \{ a_{ij} b_{kl} + a_{il} b_{jk} + a_{kl} b_{ij} + a_{jk} b_{il} - a_{ik} b_{jl} - a_{jl} b_{ik} \\
&\quad + \frac{4}{N-2} \left(-(ab)_{[ik]} \sigma_{jl} + (ab)_{[il]} \sigma_{jk} - (ab)_{[jl]} \sigma_{ik} + (ab)_{[jk]} \sigma_{il} \right. \\
&\quad \left. + (ab)_{[ij]} \sigma_{kl} + (ab)_{[kl]} \sigma_{ij} \right) + \frac{4((ab))}{(N-1)(N-2)} (\sigma_{ik} \sigma_{jl} - \sigma_{il} \sigma_{jk} - \sigma_{ij} \sigma_{kl}) \} \\
\mathcal{P}_{\mu_1 + \mu_3} &= \frac{1}{2} \left[a_{ij} b_{kl} - a_{kl} b_{ij} + \frac{4}{N-2} \left((ab)_{(il)} \sigma_{jk} + (ab)_{(jk)} \sigma_{il} \right. \right. \\
&\quad \left. \left. - (ab)_{(ik)} \sigma_{jl} - (ab)_{(jl)} \sigma_{ik} \right) \right] \\
\mathcal{P}_{2\mu_2} &= \frac{1}{6} \{ 2a_{ij} b_{kl} + 2a_{kl} b_{ij} + a_{kj} b_{il} + a_{ik} b_{jl} + a_{jl} b_{ik} + a_{il} b_{kj} \\
&\quad + \frac{1}{N-2} [(4(ab)_{ki} + 2(ab)_{ik}) \sigma_{jl} - (4(ab)_{li} + 2(ab)_{il}) \sigma_{jk} \\
&\quad + (4(ab)_{lj} + 2(ab)_{jl}) \sigma_{ik} - (4(ab)_{kj} + 2(ab)_{jk}) \sigma_{il} - 4(ab)_{[kl]} \sigma_{ij}] \\
&\quad + \frac{((ab))}{(N-1)(N-2)} (2\sigma_{ik} \sigma_{jl} - 2\sigma_{il} \sigma_{jk} + 4\sigma_{ij} \sigma_{kl}) \} \\
\mathcal{P}_{2\mu_1} &= B_{ik} \sigma_{jl} - B_{il} \sigma_{jk} + B_{jl} \sigma_{ik} - B_{jk} \sigma_{il} \\
\mathcal{P}_{\mu_2} &= C_{ik} \sigma_{jl} - C_{il} \sigma_{jk} + C_{jl} \sigma_{ik} - C_{jk} \sigma_{il} \\
\mathcal{P}_0 &= \frac{((ab))}{N(N-1)} (\sigma_{ik} \sigma_{jl} - \sigma_{jk} \sigma_{il})
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
(ab)_{ij} &= \sum_{k,l=1}^N a_{ik} b_{jl} \sigma_{kl}, \quad ((ab)) = \sum_{i,j=1}^N (ab)_{ij} \sigma_{ij}, \\
B_{ij} &= \frac{1}{2(N-2)} ((ab)_{ij} + (ab)_{ji}), \quad C_{ij} = \frac{1}{2(N-2)} \left((ab)_{ij} - (ab)_{ji} - \frac{2}{N} ((ab)) \right)
\end{aligned}$$

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